Nonlinear saturation of the turbulent α effect

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We study the saturation of the turbulent α effect in the nonlinear regime. A numerical experiment is constructed based on the full nonlinear magnetohydrodynamics equations that allows the α effect to be measured for different values of the mean magnetic field. The object is to distinguish between two possible theories of nonlinear saturation. It is found that the results are in close agreement with the theories that predict strong suppression and are incompatible with those that predict that the turbulent α effect persists up to mean fields of order of the equipartition energy.
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The purpose of dynamo theory is to explain the origin of magnetic fields in terms of the hydromagnetic conversion of kinetic energy into magnetic energy. Turbulent motions of a highly conducting fluid can stretch magnetic field lines, thereby leading to the amplification of the magnetic field [1,2]. Observationally, it is often found that cosmical magnetic fields have energy densities comparable to the kinetic energy of the ambient medium—equipartition. Furthermore, the scale of variation of these fields is often much larger than the coherence length of the underlying turbulent flow [3]. It is therefore necessary to explain not merely the process by which magnetic fields can be generated, but how large scale fields can be amplified to equipartition strength. This has motivated the large scale dynamo problem that deals precisely with the generation of magnetic fields on scales larger than the scale of variation of the velocity.

There are several mechanisms for large scale dynamo action. The most widely invoked is the α effect, which describes the generation of large scale magnetic fields through the interaction of (small scale) velocity and magnetic fluctuations in flows lacking reflectional symmetry [4]. Most of these mechanisms have been studied extensively in the kinematic approximation (linear regime) which is valid when the magnetic field is weak. However, little is known about their behavior in the nonlinear regime that eventually ensues as the magnetic field strength increases.

The α effect expresses a relationship between the average electromotive force $\mathcal{E}$ and the average magnetic field $\langle\mathbf{B}\rangle$, namely,

$$\mathcal{E}_i=\langle\mathbf{u}\times\mathbf{b}\rangle_i=\alpha_{ij}\langle\mathbf{B}\rangle_j-\beta_{ijk}\partial_j\langle\mathbf{B}\rangle_k+\cdots, \tag{1}$$

where $\mathbf{u}$ and $\mathbf{b}$ are the *fluctuating* velocity and magnetic field and the averages are taken over volumes large compared with the velocity correlation length [1]. In the kinematic regime this relationship is linear and the α tensor can be regarded as a statistical property of the advecting field of turbulence. On general dimensional grounds for helical turbulence and with magnetic Reynolds number $R_m\gg 1$, one expects α to be independent of diffusive processes — hence the term turbulent α effect—and to be given by $\alpha=\alpha_0\approx u=\langle\mathbf{u}^2\rangle^{1/2}$ [1]. In the nonlinear regime α will depend on $\langle\mathbf{B}\rangle$. Traditionally, a dependence of the form

$$\alpha=\alpha_0(1+\langle\mathbf{B}\rangle^2/u^2)^{-1} \tag{2}$$

has been assumed. According to Eq. (2) the α effect retains its kinematic (turbulent) value even when the energy of the mean field is $O(u^2)$ — equipartition [5]. This point of view has been challenged by several authors [6] who have instead proposed a dependence of the form

$$\alpha=\alpha_0(1+R_m\langle\mathbf{B}\rangle^2/u^2)^{-1}. \tag{3}$$

The presence of R_m in the denominator of Eq. (3) implies that the α effect becomes suppressed when $|\langle\mathbf{B}\rangle|^2\approx u^2/R_m$. Furthermore, according to Eq. (3), for mean fields of equipartition strength the α effect is no longer turbulent but depends on (collisional) diffusive processes. For two reasons it is important to distinguish between these alternative formulations. One is that they describe very different underlying physical processes. The other is because of the huge values of R_m prevalent in most astrophysical circumstances; if Eq. (3) is correct it implies that the α effect is nonlinearly ineffective and therefore cannot easily be invoked as a mechanism for the generation of large scale fields of equipartition strength. The objective of this paper is to present the results of a series of numerical experiments designed specifically to distinguish between these two alternatives.

The evolution of the (incompressible) velocity $\mathbf{U}$ and magnetic field $\mathbf{B}$ can be described by the (dimensionless) equations

$$(\partial_t-R_m^{-1}\nabla^2)\mathbf{B}=\nabla\times(\mathbf{U}\times\mathbf{B}), \tag{4}$$

$$(\partial_t-R_e^{-1}\nabla^2)\mathbf{U}+\mathbf{U}\cdot\nabla\mathbf{U}=-\nabla p+\mathbf{J}\times\mathbf{B}+\mathbf{F}, \tag{5}$$

$$\nabla\cdot\mathbf{B}=\nabla\cdot\mathbf{U}=0, \tag{6}$$

where p is the pressure, $\mathbf{J}=\nabla\times\mathbf{B}$ is the electric current, and R_e and R_m are the kinetic and magnetic Reynolds numbers, respectively. In terms of these equations, the kinematic regime occurs when the Lorentz force $\mathbf{J}\times\mathbf{B}$ in Eq. (5) is negligible and the velocity is determined solely by the forcing

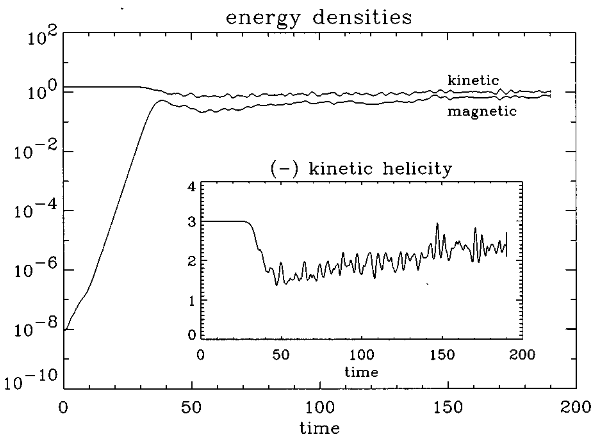

FIG. 1. Temporal evolution of the kinetic and magnetic energies and kinetic helicity (inset) for a case with $R_m = R_e = 100$ and $B_0 = 0$. The kinematic phase, during which the field grows exponentially and the back reaction is negligible, ends at $t \approx 30$.

function **F**. By contrast, in the nonlinear regime the Lorentz force is dynamically important and the velocity depends on both **F** and **B**.

We choose **F** to give a computationally convenient velocity in the kinematic regime. We let

$$\mathbf{U}_0 = (\partial_y \psi, -\partial_x \psi, \psi), \tag{7}$$

$$\psi = \sqrt{3/2}[\sin(x + \cos t) + \cos(y + \sin t)]. \tag{8}$$

and define

$$\mathbf{F} = (\partial_t - R_e^{-1} \nabla^2) \mathbf{U}_0. \tag{9}$$

Since $\nabla \cdot \mathbf{U}_0 = 0$ and $\mathbf{U}_0 \cdot \nabla \mathbf{U}_0 = \nabla \Phi$, in the absence of magnetic perturbations the forcing **F** drives the velocity $\mathbf{U}_0$ [7]. By construction $\mathbf{U}_0$ is periodic in x and y and independent of z. Its properties have been extensively studied in the context of fast dynamo action and it is known that $\mathbf{U}_0$ has large regions of chaotic streamlines. Weak magnetic perturbations of the form

$$\mathbf{B}(\mathbf{x},t) = \hat{\mathbf{B}}(x,y,t) \exp ikz \tag{10}$$

grow (initially) exponentially under the action of $\mathbf{U}_0$ [8], where the growth rate depends on k with a maximum value of ≈ 0.3 at $k = k_0 = 0.57$. Accordingly, we solve Eqs. (4)–(6) in a periodic domain of size $2\pi \times 2\pi \times 2\pi/k_0$, so that the computational domain exactly contains the fastest growing (kinematic) eigenfunction.

We assume that the magnetic field is initially of the form

$$\mathbf{B}(\mathbf{x},0) = (0,0,B_0) + \delta \mathbf{b}, \tag{11}$$

where B_0 is a constant and $\delta \mathbf{b}$ is a weak, zero-mean random perturbation. By use of the divergence theorem $\langle \mathbf{B} \rangle = B_0 \hat{\mathbf{z}}$ for all times, where hereafter the angle brackets denote an average over the computational volume.

The typical evolution of this system can be inferred from Fig. 1, which shows the time histories of the kinetic and magnetic energy densities and the kinetic helicity for a case with $B_0 = 0$. There is an initial kinematic phase where

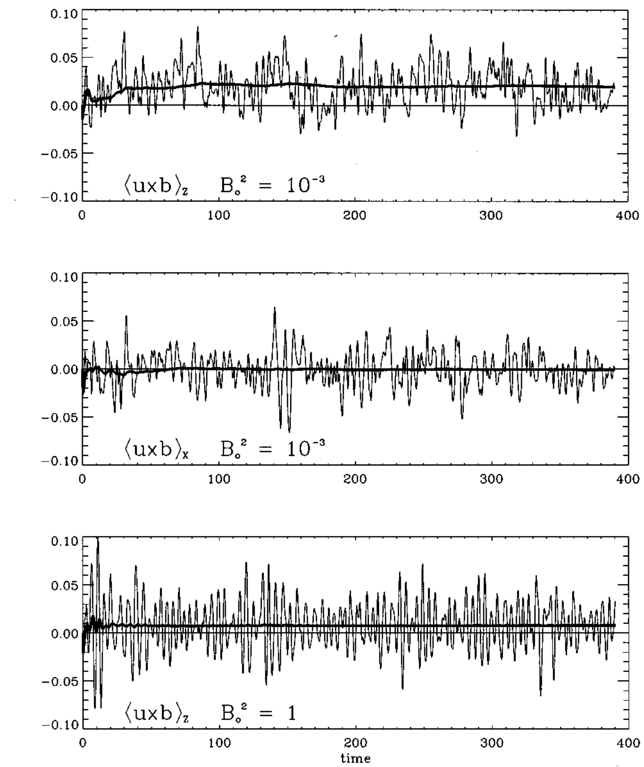

FIG. 2. Time histories and time averages (thick lines) of $\langle \mathbf{u} \times \mathbf{b} \rangle$. The two uppermost panels show the z and x components for a case with $B_0^2 = 10^{-3}$. The lower panel shows the z component for a case with $B_0^2 = 1$. Both cases have $R_m = R_e = 100$. The origin of the time axis corresponds to the beginning of the measurement period, not to the beginning of the calculation.

$\mathbf{U} \approx \mathbf{U}_0$ and the magnetic field grows exponentially. When the amplitude of the magnetic field becomes comparable with the rms velocity the Lorentz force becomes important and the exponential growth saturates. In the subsequent dynamical phase the velocity, which is still strongly helical, differs substantially from $\mathbf{U}_0$. In particular it is important to note that the velocity field is now z dependent and hence fully three dimensional. We can thus effect the decomposition

$$\mathbf{U} = \langle \mathbf{U} \rangle_V + \mathbf{u}, \quad \mathbf{B} = \langle \mathbf{B} \rangle_V + \mathbf{b}, \tag{12}$$

where $\langle \mathbf{U} \rangle_V$ and $\langle \mathbf{B} \rangle_V$ are the z-independent parts of **U** and **B**, respectively, and define the α tensor through the relation

$$\langle \mathbf{u} \times \mathbf{b} \rangle_i = \alpha_{ij}(t) \langle \mathbf{B} \rangle_j . \tag{13}$$

Clearly, since $\langle \mathbf{B} \rangle = (0,0,B_0)$ only the α_{i3} components are accessible to measurements within the present experimental setup.

The α tensor is a statistical quantity typically defined in terms of averages over volumes much larger than the velocity correlation length. The averaging is thus carried out over many (simultaneous) independent events associated with the velocity fluctuations. In the present experiment the spatial scales of the velocity fluctuations are comparable in size with the entire computational domain; we therefore expect $\alpha_{i3}(t)$ as defined above to be a strongly fluctuating quantity.

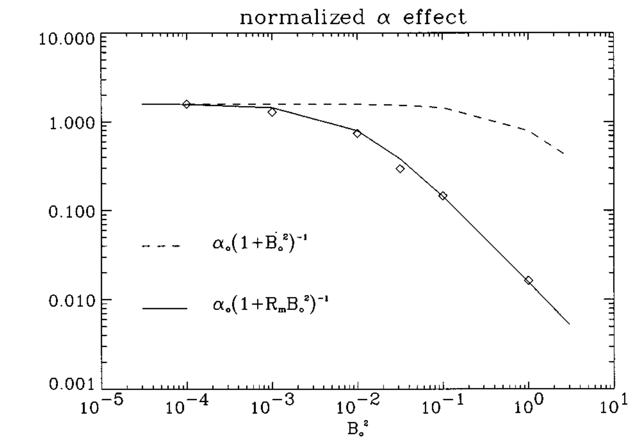

FIG. 3. Normalized α effect, $\alpha_N=\overline{\alpha_{33}}/\overline{\langle \mathbf{u}^2\rangle}$, as a function of B_0^2 for cases with $R_m=R_e=100$. Each diamond corresponds to a numerical simulation. The dashed and solid curves are fits to the data using Eqs. (2) and (3) with $R_m=100$, respectively. In both cases the value at $B_0^2=10^{-4}$ has been fitted exactly.

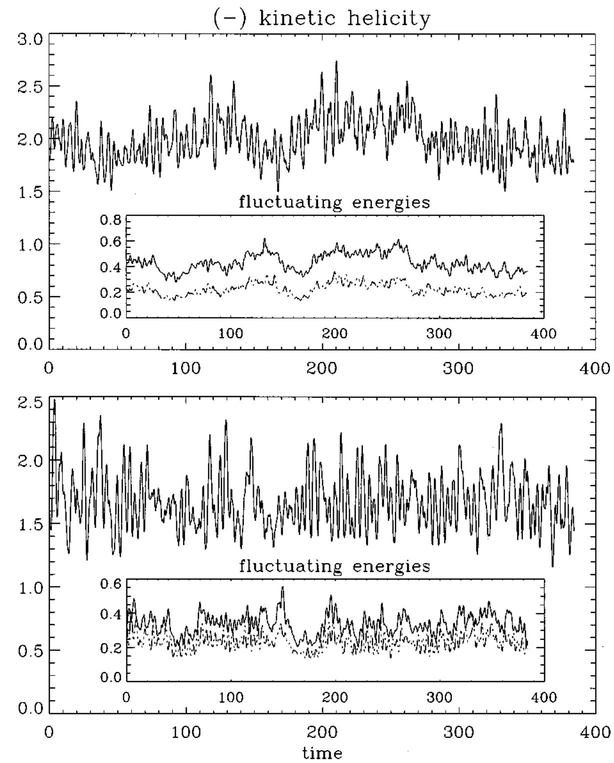

FIG. 4. Time histories of the kinetic helicity and fluctuating kinetic (solid line) and magnetic (dashed line) energy densities. The upper panel corresponds to a case with $B_0^2=10^{-3}$, the lower to a case with $B_0^2=1$. In both cases $R_m=R_e=100$. The origin of the time axis corresponds to the beginning of the measurement period, not to the beginning of the calculation.

In order to recover the statistical nature of the α effect we introduce the time average of α_{i3}

$$\overline{\alpha_{i3}}(t_2-t_1)=\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\alpha_{i3}(t)dt, \tag{14}$$

with the assumption that the turbulence is stationary and has a finite correlation time τ so that events separated in time by more than τ are once again statistically independent. Clearly this assumption has to be verified *a posteriori*.

The behavior of $\langle \mathbf{u}\times\mathbf{b}\rangle$ for some typical runs with $R_m=R_e=100$ is illustrated in Fig. 2 which shows the time histories of the x and z components of $\langle \mathbf{u}\times\mathbf{b}\rangle$ for a case with $B_0^2=10^{-3}$ and the z component for a case with $B_0^2=1$. The superposed thick lines show the time averages up to time t. It is clear that even though $\langle \mathbf{u}\times\mathbf{b}\rangle$ is a strongly fluctuating quantity its time average is well defined and can therefore be used to compute $\overline{\alpha}$. From symmetry considerations we expect $\overline{\alpha_{13}}$ and $\overline{\alpha_{23}}$ to vanish since the mean field is in the z direction. This is confirmed by the time traces in the central panel in Fig. 2 which show that indeed the x component of $\overline{\langle \mathbf{u}\times\mathbf{b}\rangle}\rightarrow 0$ as the length of the averaging becomes large.

We have computed a number of cases with $R_m=R_e=100$ and $10^{-4}\leqslant B_0^2\leqslant 1$. The calulations were carried out using a fully dealiased, pseudospectral code (see [9] for details). The results are summarized in Fig. 3 where we display the *normalized* α effect, $\alpha_N=\overline{\alpha_{33}}/\overline{\langle \mathbf{u}^2\rangle}$. The solid and dashed curves are given by expressions (3) and (2), respectively. In both cases we have simply fitted the value at $B_0^2=10^{-4}$ exactly, and for the solid curve we have used $R_m=100$. It is clear that the data are in very good agreement with Eq. (3) and incompatible with Eq. (2). The conclusion to be drawn here is that, at least within the framework of the present experiment, the *turbulent* α effect indeed becomes strongly suppressed when the squared mean field exceeds a critical value $B_{\rm crit}^2\approx u^2/R_m$. Furthermore, the α effect becomes a diffusive process as B_0^2 approaches u^2.

It is useful to search for the physical basis for this result. In order to make progress in that direction we note the following points. A key feature of the present experiment is that the α effect is *nonlinearly driven*. In order to appreciate the significance of this it is helpful to distinguish between two separate kinematic regimes in addition to the fully dynamical regime. First there is a true kinematic regime in which the Lorentz force is negligible on all scales and the magnetic evolution is characterized by the exponential growth of the fastest growing eigenfunction. This corresponds to the period $t\leqslant 30$ in Fig. 1. In this phase there is *no* α effect as we have defined it, since the velocity $\approx\mathbf{U}_0$, which is z independent; i.e., there are no velocity fluctuations. At later times the velocity has been modified by the action of the Lorentz force and contains fluctuations on scales comparable to or smaller than the vertical extent of the fastest growing eigenfunction. It is these fluctuations that can give rise to the α effect. We may now define an intermediate regime in which the velocity is fully dynamical with respect to the small scale magnetic fluctuations, which are of order the equipartition value, but still kinematic with respect to the action of the mean field. In this regime increasing the amplitude of the mean field does not greatly affect the value of the α effect. For example, in going from $B_0^2=10^{-4}$ to $B_0^2=10^{-2}$, the α effect only changes by a factor of 2 (see Fig. 3). For larger values of the mean field the velocity is dynamical with respect to both the small scale and large scale components of the magnetic field. In this fully dynamical regime the turbulent α effect strongly decreases with increasing B_0^2 according to Eq. (3).

It would be erroneous to assume that this strong decrease of the α effect as $B_0^2 \rightarrow 1$ is due to a corresponding reduction in the vigor of the turbulent fluctuations. This possibility can be ruled out by inspection of Fig. 4, which shows the time histories of the kinetic helicity and fluctuating energies for two cases with $B_0^2 = 10^{-3}$ and $B_0^2 = 1$. Clearly there is hardly any difference in helicity and turbulent excitation level in these two cases that nevertheless differ in efficiency of the α effect by a factor of nearly 100 (R_m actually). The conclusion is that the reduction in the α effect is due to a change in the character of the velocity fluctuations and not in their amplitude. It should be noted that the suppression of the α effect described by Eq. (3) is entirely analogous to the suppression of the turbulent diffusivity in two-dimensional magnetohydrodynamic turbulence [10]. For that case it is known that the suppression of turbulent diffusivity is associated with the development of a long term memory by the turbulence [11]. When $B_0^2 = 1$ the velocity fluctuations are strongly influenced by Alfvén waves that, over successive cycles, distort the mean field first one way and then the other without giving rise to any net distortion. In other words the field lines *remember* their initial position except for the effect of (collisional) diffusion. This small diffusive effect manifests itself by the factor of R_m in the denominator of Eq. (3).

Finally we note that if the conclusions of the present paper apply to astrophysical situations, in other words, if Eq. (3) holds even for very large values of R_m, then the α effect will be ineffective and simply incapable of generating sizable mean fields. Consequently, models of magnetic field generation based on a straightforward application of the turbulent α effect will have to be substantially modified or abandoned altogether.

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